College – S. S. College, J. Bad

Deptt – Mathematics

Topic – {Problem and Certain Theorems Based On Euler's Theorem Contd}

Class – B.Sc I (Hons)

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By Samrendra Kumar

(Part-time Teacher)

Problem based on Euler's theorem (partial differentiation)

Problem 5.

$u = \sin^{-1} \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ show that

$$\frac{\partial u}{\partial x} = -\frac{y}{x} \frac{\partial u}{\partial y}$$

Solution ⇒ By the question $u = \sin^{-1} \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

$$\Rightarrow \sin u = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

Let $v = \sin u = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

clearly $\sin u = v$ is a homogeneous function of x and y of zero order.

⇒ By Euler's theorem

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 0 \quad \text{———} \quad (1)$$

$v = \sin u$

$$\frac{\partial v}{\partial x} = \frac{\partial \sin u}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$= \cos u \frac{\partial u}{\partial x}$$

and $\frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}$

$\therefore$ From (1)

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow x \cos u \frac{\partial u}{\partial x} = -y \cos u \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$$ proved

Problem 6

If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$ prove that

$$\frac{x\partial u}{\partial x} + \frac{y\partial u}{\partial y} = \frac{1}{2}\tan u$$

Solution :→ By the question

$$u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Say $v = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Clearly $v = \sin u$ = is the Homogeneous function of x and y of order $1/2$

$$\therefore \frac{x\partial v}{\partial x} + \frac{y\partial v}{\partial y} = \frac{1}{2}v \quad \text{——— (1)}$$

(By Euler's theorem)

$v = \sin u \Rightarrow \frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}$

and $\frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}$

putting these values in (1)

$x \cdot \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \frac{\sin u}{\cos u} = \frac{1}{2} \tan u$ proved

Problem 7 :→ If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$ prove that

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

Solution By the question

$u = \tan^{-1} \frac{x^3 + y^3}{x - y}$

$\Rightarrow \tan u = \frac{x^3 + y^3}{x - y}$

Say $v = \tan u = \frac{x^3 + y^3}{x - y}$

clearly $v = \tan u$ is the Homogeneous function of x and y of order 2.

hen By Euler's Theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v \quad \text{——— (1)}$$

Here $v = \tan u$

$$\Rightarrow \frac{\partial v}{\partial x} = \sec^2 u \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

Now From (1)

$$x\sec^2 u \frac{\partial u}{\partial x} + y\sec^2 u \frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u} \times \frac{1}{\sec^2 u} = \frac{2\sin u}{\cos u \cdot \sec u \cdot \sec u}$$

($\sin 2\theta = 2\sin\theta\cos\theta$)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$= \sin 2u \quad \text{proved}$$

Problem:→8 If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}(y/x)$

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution. Here $u = \sin^{-1}\frac{x}{y} + \tan^{-1} y/x$

$$= \tan^{-1}\frac{x}{\sqrt{y^2-x^2}} + \tan^{-1} y/x$$

Here $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

$$= \cancel{\sin^{-1}}\ \tan^{-1}\frac{x}{\sqrt{y^2-x^2}} + \tan^{-1} y/x$$

$$= \tan^{-1}\frac{\frac{x}{\sqrt{y^2-x^2}} + \frac{y}{x}}{1 - \frac{x}{\sqrt{y^2-x^2}}\cdot\frac{y}{x}}$$

[using $\tan^{-1}x$ ... $= \tan^{-1}\frac{x+ ...}{1-x ...}$

$$= \tan^{-1}\frac{x^2 + y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2} - xy}$$

$$\Rightarrow \tan u = \frac{x^2 + y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2} - xy}$$

Clearly $\tan u$ is the Homogeneous function of x and y of order ze

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0 \quad (1) \quad \text{where } v = \tan u$$

$$\frac{\partial v}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

Putting these values in (1)

$$x\sec^2 u \frac{\partial u}{\partial x} + y\sec^2 u \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad \text{proved}$$

<u>Problem</u> 9 :- If $u = \cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

<u>Solution</u> :-> By the question

$$u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$\Rightarrow \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ clearly, $v = \cos u$ is here homogeneous function of x and y of order $\frac{1}{2}$.

∴ By Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}v$$

Here $v = \cos u$

$$\frac{\partial v}{\partial x} = -\sin u\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = -\sin u\frac{\partial u}{\partial y}$$

$$\Rightarrow -x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u \;\therefore\; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

proved

Show that If $u = f(x,y)$ is a homogenous Function of degree n

then

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Proof :→ Here $u = f(x,y)$ is a homogenous Function of x and y of order n

then From Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{——— (1)}$$

Diff. (1) partially with respect to x

$$\frac{\partial u}{\partial x} + x\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$$ (i) is proved

Again Diff. (1) partially with respect to y

$$\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)=n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$$ (111) is proved

$x \times$ (11) $+ y \times$ (111)

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$=(n-1)u$$

Proved

Problem 1- If $u=\dfrac{(x^2+y^2)^m}{2m(m-1)}+x\phi(y/x)+\psi(y/x)$

Show that-

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)^m$$

By the question, we have

$$u = \frac{(x^2+y^2)^m}{2m(2m-1)} + x\,\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right), \text{ then}$$

We are to show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (x^2+y^2)^m$$

Let $X = \frac{(x^2+y^2)^m}{2m(2m-1)}$ $\quad Y = x\,\phi(y/x)$ $\quad Z = \psi(y/x)$

Here X is Homogeneous Function of x and y of degree ~~order~~ $2m$, Y is Homogeneous Function of order degree 1st and Z is homogeneous Function of zero degree

Hence By Euler's theorem

$$x^2\frac{\partial^2 X}{\partial x^2} + 2xy\frac{\partial^2 X}{\partial x \partial y} + y^2\frac{\partial^2 X}{\partial y^2} = 2m(2m-1)X \quad \text{—(i)}$$

$$x^2\frac{\partial^2 Y}{\partial x^2} + 2xy\frac{\partial^2 Y}{\partial x \partial y} + y^2\frac{\partial^2 Y}{\partial y^2} = 1(1-1)Y = 0 \quad \text{—(ii)}$$

$$x^2\frac{\partial^2 Z}{\partial x^2} + 2xy\frac{\partial^2 Z}{\partial x \partial y} + y^2\frac{\partial^2 Z}{\partial y^2} = 0(0-1)Z = 0 \quad \text{—(iii)}$$

Were $u = x + y + z$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 x}{\partial x \partial y} + \frac{\partial^2 y}{\partial x \partial y} + \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 z}{\partial y^2}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2m(2m-1)x$$

$$= 2m(2m-1) \frac{(x^2+y^2)^m}{2m(2m-1)} = (x^2+y^2)^m.$$

Proved

Problems. If $u = \operatorname{cosec}^{-1} \sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}$ show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$$

Sol^n

By the question

$$u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$\Rightarrow \operatorname{cosec} u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$\Rightarrow \operatorname{cosec}^2 u = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}$$

$$\Rightarrow \sin^2 u = \frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}} = v \text{ (Say)}$$

From Above, As it is clear that v is Homogeneous Function of x and y of degree $\frac{1}{3}-\frac{1}{2}=\frac{-1}{6}$

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{-1}{6}v \quad \text{——— (1)}$$

$$v = \sin^2 u$$

$$\frac{\partial v}{\partial x} = 2\sin u \cos u \frac{\partial u}{\partial x} = \sin 2u \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \sin 2u \frac{\partial u}{\partial y}$$

Hence From (1)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}\sin^2 u \quad \text{——— (A)}$$

Also $x^2\dfrac{\partial^2 v}{\partial x^2} + 2xy\dfrac{\partial^2 v}{\partial x\partial y} + y^2\dfrac{\partial^2 v}{\partial y^2}$

$$= -\frac{1}{6}\left(-\frac{1}{6}-1\right)v$$

$$= \frac{1}{6}\left(\frac{7}{6}\right)v$$

$$= \frac{7}{36}v = \frac{7}{36}\sin^2 u \quad \text{——— (B)}$$

$\therefore\ v = \sin^2 u$

$$\frac{\partial v}{\partial x} = \sin 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = 2\cos 2u\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \sin 2u\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial y^2} = 2\cos u\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial y} + \sin 2u\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\sin 2u\frac{\partial u}{\partial x}\right)$$

$$= \sin 2u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}2\cos 2u\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \sin 2u \frac{\partial^2 u}{\partial x \partial y} + 2\cos 2u \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}$$

Hence From (A)

$$x \sin 2u \frac{\partial u}{\partial x} + y \sin 2u \frac{\partial u}{\partial y} = -\frac{1}{6} \sin^2 u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{6} \frac{\sin^2 u}{\sin 2u}$$

$$= -\frac{1}{6} \frac{\sin^2 u}{2\sin u \cos u}$$

$$= -\frac{1}{6} \tan u \text{. ——— (B)}$$

From B

$$x^2 \left[2\cos 2u \left(\frac{\partial u}{\partial x}\right)^2 + \sin 2u \frac{\partial^2 u}{\partial x^2}\right) + 2xy \left[\sin 2u \frac{\partial^2 u}{\partial x \partial y} + 2\cos 2u \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}\right] + y^2 \left[2\cos 2u \left(\frac{\partial u}{\partial y}\right)^2 + \sin 2u \frac{\partial^2 u}{\partial y^2}\right]$$

$$= \frac{7}{36} \sin^2 u$$

$$\Rightarrow \left\{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right\}\sin 2u$$

$$+ 2\cos 2u\left\{x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2\right\}$$

$$= \frac{7}{36}\sin^2 u$$

$$\Rightarrow \left[x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right]\sin 2u$$

$$= \frac{7}{36}\sin^2 u - 2\cos 2u\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\}^2$$

$$= \frac{7}{36}\sin^2 u - 2\cos 2u\left\{-\frac{1}{12}\tan u\right\}^2$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{7}{36}\frac{\sin^2 u}{\sin 2u} - \frac{2\cos 2u}{\sin 2u}\cdot\frac{1}{144}\tan^2 u$$

$$= \frac{7}{36}\frac{\sin^2 u}{2\sin u\cos u} - \frac{2}{\tan 2u}\cdot\frac{1}{144}\tan^2 u$$

$$= \frac{7}{36}\times\frac{1}{2}\tan u - \frac{1}{72}\frac{1-\tan^2 u}{2\tan u}\cdot\tan^2 u$$

$$= \frac{7}{72}\tan u - \frac{1}{72}\tan u\,(1-\tan^2 u)$$

$$= \frac{1}{72}\tan u\left[7 - \frac{1-\tan^2 u}{2}\right] = \frac{1}{72}\left[\frac{13+\tan^2 u}{2}\right]$$

Hence Result